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A LAYERED COMPOSITE WITH A BROKEN LAMINATE†

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Abstract—The problem of a laminate composite in presence of a crack located normal to the bond lines is considered. Stress analysis of the limiting case when the crack extends to the bond lines is carried out. Integral transform technique is used to formulate the problem in terms of a singular integral equation from which the power of stress singularity around the crack tip terminating at the interface is obtained. The singular integral equation is solved numerically and the effect of material properties on the stress intensity factor is calculated.

INTRODUCTION

The problem of a laminate composite with a crack normal to the interfaces was studied by Hilton and Sih [1]. The composite geometry in [1] consisted of a cracked layer bonded between two half-planes of different elastic properties. This is an idealization of a many-layered composite where one concentrates on a single layer and approximates the effect of outer layers by prescribing some average elastic properties to the half-planes.

The aim of this study is to reconsider the same problem and analyze the case when the crack has propagated through the layer and is just touching the interfaces. The limiting cases when the matrix is rigid or has zero modulus of rigidity correspond to that of a semi-infinite strip with constrained or free sides respectively. The method of solution uses the displacement expressions derived by Sneddon [2] where the problem of an infinite strip with a central crack normal to the strip boundaries is solved. Using this integral transform technique, first a crack problem will be formulated in terms of dual integral equations which will then be reduced to give a singular integral equation. This standard technique has been previously used by Erdogan and Gupta in various papers concerning crack problems [3–5] in composite structures. For the case of a crack away from the interface, the only singular kernel appearing in the integral equation is a Cauchy kernel. Hilton and Sih [1] have treated this problem using the technique followed by Sneddon [2], where it is possible to reduce the problem to a Fredholm equation of second kind. This method, however, is not applicable when the crack extends to the interface. The singular integral equation in that case will be shown to contain a set of generalized Cauchy kernels also. The final integral equation is then solved by using Gauss–Jacobi integration technique [6, 7]. The effect of material properties on the stress intensity factor at the crack tip is presented both graphically and in tabular form.

Recently the same problem has been considered by Ash baugh [8]. The formulation of the problem in [8] is similar to that given here. A singular differential–integral equation is finally obtained, which is numerically solved by expanding the unknown function in terms

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of an infinite series of ultra-spherical polynomials multiplied by their weighting function. The procedure, though extremely cumbersome, nevertheless gives satisfactory results. The results obtained in [8] will be compared with those in this paper.

FORMULATION OF PROBLEM

Consider a laminate composite in plane strain conditions consisting of a single layer of width $2h$, shear modulus μ_1 and Poisson's ratio ν_1 bonded to two half-planes having shear modulus and Poisson's ratio as μ_2 and ν_2, respectively (Fig. 1). A crack of length

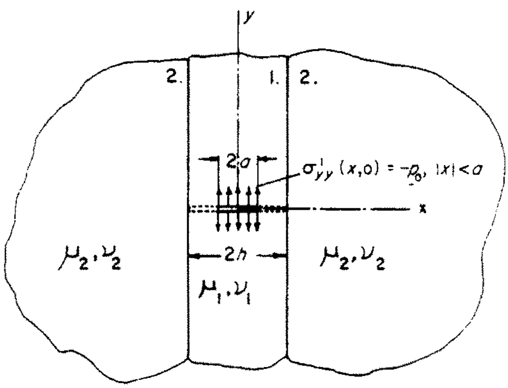

FIG. 1. Geometry of the laminate composite.

$2a$ ($a \leq h$) is located centrally along the x-axis. Only the symmetric problem will be solved in this paper. The skew-symmetric case can be treated in an identical manner. Subscripts or superscripts 1 and 2 will be used to refer to the layer and the half-planes, respectively.

Considering symmetric normal tractions on the crack surface being the only loads [1], we have the following boundary and continuity conditions.

Continuity conditions at $x = h$

$$\begin{aligned} u_1(h, y) &= u_2(h, y); \qquad v_1(h, y) = v_2(h, y) \\ \sigma^1_{xx}(h, y) &= \sigma^2_{xx}(h, y); \qquad \sigma^1_{xy}(h, y) = \sigma^2_{xy}(h, y). \end{aligned} \tag{1}$$

Homogeneous conditions at $y = 0$

$$\begin{aligned} &\sigma^1_{xy}(x, 0) = 0, \quad |x| < h; \qquad \sigma^2_{xy}(x, 0) = 0, \quad |x| > h \\ &v_2(x, 0) = 0, \quad |x| > h. \end{aligned} \tag{2}$$

Mixed boundary conditions at $y = 0$

$$\begin{aligned} &\sigma^1_{yy}(x, 0) = -p(x), \quad |x| < a \\ &v_1(x, 0) = 0, \quad a < |x| < h. \end{aligned} \tag{3}$$

The displacement field for the layer as derived by Sneddon [2] is a superposition of well-known transform solutions [3] for a body with $x = 0$ and $y = 0$ as planes of symmetry and an upper half-plane symmetrical about the y-axis. It may be expressed as

$$\begin{aligned} u_1(x, y) &= -\frac{2}{\pi}\int_0^\infty \left\{\frac{1}{\eta}\left[f_1(\eta)-\frac{\kappa_1-1}{2}g_1(\eta)\right]\sinh(\eta x)+xg_1(\eta)\cosh(\eta x)\right\}\cosh\eta y\,d\eta \\ &\quad -\frac{2}{\pi}\int_0^\infty \frac{\phi_1(\xi)}{\xi}\left(\frac{\kappa_1-1}{2}-\xi y\right)e^{-\xi y}\sin\xi x\,d\xi \\ v_1(x, y) &= \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{\eta}\left[f_1(\eta)+\frac{\kappa_1+1}{2}g_1(\eta)\right]\cosh(\eta x)+xg_1(\eta)\sinh(\eta x)\right\}\sin\eta y\,d\eta \\ &\quad +\frac{2}{\pi}\int_0^\infty \frac{\phi_1(\xi)}{\xi}\left(\frac{\kappa_1+1}{2}+\xi y\right)e^{-\xi y}\cos\xi x\,d\xi \end{aligned} \tag{4}$$

where

$$\kappa_i = 3-4v_i.$$

The corresponding stress field is given by the equations

$$\begin{aligned} \frac{\sigma_{xx}^1(x, y)}{2\mu_1} &= -\frac{2}{\pi}\int_0^\infty [f_1(\eta)\cosh(\eta x)+\eta xg_1(\eta)\sinh(\eta x)]\cos\eta y\,d\eta \\ &\quad -\frac{2}{\pi}\int_0^\infty \phi_1(\xi)(1-\xi y)e^{-\xi y}\cos\xi x\,d\xi \\ \frac{\sigma_{yy}^1(x, y)}{2\mu_1} &= \frac{2}{\pi}\int_0^\infty \{[f_1(\eta)+2g_1(\eta)]\cosh(\eta x)+\eta xg_1(\eta)\sinh(\eta x)\}\cos\eta y\,d\eta \\ &\quad -\frac{2}{\pi}\int_0^\infty \phi_1(\xi)(1+\xi y)e^{-\xi y}\cos\xi x\,d\xi \\ \frac{\sigma_{xy}^1(x, y)}{2\mu_1} &= \frac{2}{\pi}\int_0^\infty \{[f_1(\eta)+g_1(\eta)]\sinh(\eta x)+\eta xg_1(\eta)\cosh(\eta x)\}\sin\eta y\,d\eta \\ &\quad -\frac{2}{\pi}\int_0^\infty \xi y\phi_1(\xi)e^{-\xi y}\sin\xi x\,d\xi. \end{aligned} \tag{5}$$

Condition $\sigma_{xy}^1(x, 0) = 0$ is identically satisfied by this representation.

Similarly, displacement and stress fields for the half-plane can be expressed as

$$\begin{aligned} u_2(x, y) &= \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{\eta}\left[f_2(\eta)+\frac{\kappa_2-1}{2}g_2(\eta)\right]+xg_2(\eta)\right\}e^{-\eta x}\cos\eta y\,d\eta \\ v_2(x, y) &= \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{\eta}\left[f_2(\eta)-\frac{\kappa_2+1}{2}g_2(\eta)\right]+xg_2(\eta)\right\}e^{-\eta x}\sin\eta y\,d\eta \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\sigma_{xx}^2(x, y)}{2\mu_2} &= -\frac{2}{\pi}\int_0^\infty [f_2(\eta)+\eta xg_2(\eta)]e^{-\eta x}\cos\eta y\,d\eta \\ \frac{\sigma_{yy}^2(x, y)}{2\mu_2} &= \frac{2}{\pi}\int_0^\infty [f_2(\eta)+(\eta x-2)g_2(\eta)]e^{-\eta x}\cos\eta y\,d\eta \\ \frac{\sigma_{xy}^2(x, y)}{2\mu_2} &= -\frac{2}{\pi}\int_0^\infty [f_2(\eta)+(\eta x-1)g_2(\eta)]e^{-\eta x}\sin\eta y\,d\eta. \end{aligned} \tag{7}$$

Again this representation identically satisfies the homogeneous boundary conditions (2). The unknowns f_1, g_1, f_2, g_2 and ϕ_1 will be solved by using four continuity conditions (1) and the mixed boundary conditions (3). Continuity conditions (1) may be written as

$$\left[f_2(\eta)+\left(\frac{\kappa_2-1}{2}+\eta h\right)g_2(\eta)\right]e^{-\eta h} = -\left[f_1(\eta)-\frac{\kappa_1-1}{2}g_1(\eta)\right]\sinh(\eta h)-\eta h g_1(\eta)\cosh(\eta h)$$
$$-\frac{2\eta}{\pi}\int_0^\infty \frac{\phi_1(\xi)}{(\xi^2+\eta^2)^2}\left[2\eta^2+\frac{\kappa_1-3}{2}(\eta^2+\xi^2)\right]\sin \xi h\, d\xi$$

$$\left[f_2(\eta)-\left(\frac{\kappa_2+1}{2}-\eta h\right)g_2(\eta)\right]e^{-\eta h} = \left[f_1(\eta)+\frac{\kappa_1+1}{2}g_1(\eta)\right]\cosh(\eta h)+\eta h g_1(\eta)\sinh(\eta h)$$
$$+\frac{2\eta^2}{\pi}\int_0^\infty \frac{\phi_1(\xi)}{\xi(\xi^2+\eta^2)^2}\left[2\xi^2+\frac{\kappa_1+1}{2}(\eta^2+\xi^2)\right]\cos \xi h\, d\xi \tag{8}$$

$$\frac{\mu_2}{\mu_1}[f_2(\eta)+\eta h g_2(\eta)]\,e^{-\eta h} = f_1(\eta)\cosh(\eta h)+\eta h g_1(\eta)\sinh(\eta h)+\frac{4\eta^2}{\pi}\int_0^\infty \frac{\xi\phi_1(\xi)}{(\xi^2+\eta^2)^2}\cos \xi h\, d\xi$$

$$-\frac{\mu_2}{\mu_1}[f_2(\eta)+(\eta h-1)g_2(\eta)]\,e^{-\eta h} = [f_1(\eta)+g_1(\eta)]\sinh(\eta h)$$
$$+\eta h g_1(\eta)\cosh(\eta h)-\frac{4\eta}{\pi}\int_0^\infty \frac{\xi^2\phi_1(\xi)}{(\xi^2+\eta^2)^2}\sin \xi h\, d\xi.$$

Mixed boundary conditions (3) can be expressed as

$$\frac{\sigma^1_{yy}(x,0)}{2\mu_1} = \lim_{y\to 0^+} -\frac{2}{\pi}\int_0^\infty \phi_1(\xi)(1+\xi y)\,e^{-\xi y}\cos \xi x\, d\xi+\frac{2}{\pi}\int_0^\infty \{[f_1(\eta)+2g_1(\eta)]\cosh(\eta x)$$
$$+nxg_1(\eta)\sinh(\eta x)\}\, d\eta = -\frac{p(x)}{2\mu_1}, \qquad |x|<a \tag{9}$$

$$\frac{\partial v_1}{\partial x}(x,0) = -\frac{\kappa_1+1}{2}\,\frac{2}{\pi}\int_0^\infty \phi_1(\xi)\sin \xi x\, d\xi = 0, \qquad a<|x|<h. \tag{10}$$

Note that in (10) the boundary condition for displacement derivative instead of displacement* is used. This is done to have same dimensions of the equations (9) and (10).

In order to reduce the dual integral equations (9) and (10) to a singular integral equation, we define a new unknown function $G(x)$ as follows.

$$\frac{\partial v_1}{\partial x}(x,0) = G(x), \qquad |x|<h. \tag{11}$$

From (10), it is clear that $G(x) = 0$ for $a<|x|<h$. Inverting (10) and using (11), we obtain

$$-\frac{\kappa_1+1}{2}\phi_1(\xi) = \int_0^a G(t)\sin \xi t\, dt. \tag{12}$$

* Displacement v_1 is half the crack opening displacement.

From (8), eliminating f_2 and g_2, we find

$$\begin{aligned} f_1(\eta)\left(\kappa_2+\frac{\mu_2}{\mu_1}\right)\mathrm{e}^{\eta h}+g_1(\eta)\left[\left(\kappa_2+\frac{\mu_2}{\mu_1}\right)\eta h\,\mathrm{e}^{\eta h}+\kappa_2\sinh(\eta h)+\frac{\mu_2}{2\mu_1}(\mathrm{e}^{\eta h}+\kappa_1\,\mathrm{e}^{-\eta h})\right] &= E_1(\eta) \\ f_1(\eta)\left(\frac{\mu_2}{\mu_1}-1\right)\mathrm{e}^{-\eta h}+g_1(\eta)\left[\sinh(\eta h)-\left(\frac{\mu_2}{\mu_1}-1\right)\eta h\,\mathrm{e}^{-\eta h}+\frac{\mu_2}{2\mu_1}(\kappa_1\,\mathrm{e}^{\eta h}+e^{-\eta h})\right] &= E_2(\eta) \end{aligned} \tag{13}$$

where

$$\begin{aligned} E_1(\eta) &= -\frac{\mu_2}{\mu_1}[D_3(\eta)+D_4(\eta)]-\kappa_2[D_1(\eta)-D_2(\eta)] \\ E_2(\eta) &= \frac{\mu_2}{\mu_1}[D_3(\eta)-D_4(\eta)]+[D_1(\eta)+D_2(\eta)] \end{aligned} \tag{14}$$

and

$$\begin{aligned} D_1(\eta) &= \frac{4\eta^2}{\pi}\int_0^\infty \frac{\xi\phi_1(\xi)}{(\xi^2+\eta^2)^2}\cos\xi h\,\mathrm{d}\xi \\ D_2(\eta) &= \frac{4\eta}{\pi}\int_0^\infty \frac{\xi^2\phi_1(\xi)}{(\xi^2+\eta^2)^2}\sin\xi h\,\mathrm{d}\xi \\ D_3(\eta) &= \frac{2\eta}{\pi}\int_0^\infty \frac{\phi_1(\xi)}{(\xi^2+\eta^2)^2}\left[2\eta^2+\frac{\kappa_1-3}{2}(\eta^2+\xi^2)\right]\sin\xi h\,\mathrm{d}\xi \\ D_4(\eta) &= \frac{2\eta^2}{\pi}\int_0^\infty \frac{\phi_1(\xi)}{\xi(\xi^2+\eta^2)^2}\left[2\xi^2+\frac{\kappa_1+1}{2}(\eta^2+\xi^2)\right]\cos\xi h\,\mathrm{d}\xi \end{aligned} \tag{15}$$

(13) must now be solved for f_1 and g_1 and substituted in (9) to obtain an integral equation in ϕ_1, or, by using (12), for $G(x)$. Combining (12) with the first integral of (9), a simple Cauchy kernel is obtained [3–4]. Second integral in (9) would involve integrals in (15) which, when combined with (12), give the following results.

$$\begin{aligned} (\kappa_1+1)D_1(\eta) &= \int_{-a}^{a} G(t)\eta(h-t)\,\mathrm{e}^{-\eta(h-t)}\,\mathrm{d}t \\ (\kappa_1+1)D_2(\eta) &= \int_{-a}^{a} G(t)[\eta(h-t)-1]\,\mathrm{e}^{-\eta(h-t)}\,\mathrm{d}t \\ (\kappa_1+1)D_3(\eta) &= -\int_{-a}^{a} G(t)\left[\eta(h-t)+\frac{\kappa_1-1}{2}\right]\mathrm{e}^{-\eta(h-t)}\,\mathrm{d}t \\ (\kappa_1+1)D_4(\eta) &= \int_{-a}^{a} G(t)\left[\eta(h-t)-\frac{\kappa_1+1}{2}\right]\mathrm{e}^{-\eta(h-t)}\,\mathrm{d}t+\int_{-a}^{a} G(t)\,\mathrm{d}t \end{aligned} \tag{16}$$

$G(t)$ being an odd function in t makes the second term in $D_4(\eta)$ vanish. The details of above reduction may be found in Appendix A.

Using (16), the singular integral equation from (9) can be expressed as

$$\int_{-a}^{a}\frac{G(t)}{t-x}\,\mathrm{d}t+\int_{-a}^{a}G(t)K(t,x)\,\mathrm{d}t = -\frac{\pi p(x)(1+\kappa_1)}{4\mu_1}, \qquad |x|<a \tag{17}$$

where

$$K(t, x) = \int_0^\infty k(t, x, \eta)\, e^{-\eta(h-t)}\, d\eta$$

$$k(t, x, \eta) = \frac{e^{-\eta h}}{[1-4\lambda_2 \eta h\, e^{-2\eta h} - \lambda_1 \lambda_2\, e^{-4\eta h}]} \cdot [\lambda_1 \cosh(\eta x)$$
$$\{1-\lambda_2(3+2\eta h)\, e^{-2\eta h}\} - 2\eta x \lambda_1 \lambda_2 \sinh(\eta x)\, e^{-2\eta h}$$
$$+\lambda_2\{1-2\eta(h-t)\}\{\cosh(\eta x)(3-2\eta h-\lambda_1\, e^{-2\eta h})+2\eta x \sinh(\eta x)\}] \tag{18}$$
$$\lambda_1 = \frac{\kappa_1 \mu_2 - \kappa_2 \mu_1}{\mu_2 + \kappa_2 \mu_1}$$
$$\lambda_2 = \frac{\mu_2 - \mu_1}{\mu_1 + \kappa_1 \mu_2}.$$

Kernel $K(t, x)$ is a Fredholm's kernel for $a < h$ as it is bounded for all values of t and x in $(-a, a)$ is $a < h$. The singular integral equation (17) is equivalent to the final Fredholm equation obtained in [1]. The present formulation of the problem in terms of a singular integral equation has various advantages. A very simple numerical method is available to solve the equation [6]. However, the most important advantage of the procedure is that the case when $a = h$ can be studied very effectively. The integral equation (17) must be solved subject to the following single-valuedness condition.

$$\int_{-a}^{a} G(t)\, dt = 0. \tag{19}$$

The case of $a = h$

Referring to Fig. 1, when the crack extends to the interfaces (case of a broken laminate), the Fredholm kernel $K(t, x)$ in (17) is no longer bounded and contains point singularities at $t = h$ and $x = \pm h$. To extract these singularities, we need to study the infinite integral in the expression for $K(t, x)$. It turns out that the asymptotic value of the integrand $k(t, x, \eta)$ as $n \to \infty$ gives rise to these singularities. The part of the kernel contributing these singularities may be expressed as

$$K_s(t, x) = \int_0^\infty k_\infty(t, x, \eta)\, e^{-\eta(h-t)}\, d\eta \tag{20}$$

where, from (18), k_∞ is found to be

$$k_\infty(t, x, \eta) = e^{-\eta h}[\lambda_1 \cosh(\eta x) + \lambda_2\{1-2\eta(h-t)\}$$
$$\{(3-2\eta h)\cosh(\eta x)+2\eta x \sinh(\eta x)\}]. \tag{21}$$

Using the following result [9]

$$\int_0^\infty \eta^m e^{-\eta(2h-t)} \begin{Bmatrix} \sinh(\eta x) \\ \cosh(\eta x) \end{Bmatrix} d\eta = \frac{d^m}{dt^m} \int_0^\infty e^{-\eta(2h-t)} \begin{Bmatrix} \sinh(\eta x) \\ \cosh(\eta x) \end{Bmatrix} d\eta$$
$$= \frac{d^m}{dt^m}\left[\frac{1}{(2h-t)^2 - x^2} \begin{Bmatrix} x \\ 2h-t \end{Bmatrix}\right] \tag{22}$$

the singular kernel $K_s(t, x)$ may be written as

$$K_s(t,x) = (\lambda_1+3\lambda_2)\frac{2h-t}{(2h-t)^2-x^2}+\frac{2\lambda_2}{[(2h-t)^2-x^2]^2}\left[x^2t-(2h-t)^2(4h-3t)\right.$$
$$\left.+\frac{4(h-t)\{(2h-t)^2(2h^2-ht-3x^2)+x^2(6h^2-3ht-x^2)\}}{\{(2h-t)^2-x^2\}}\right]. \tag{23}$$

To analyze the behavior of the unknown function $G(t)$ near the end points, we must consider the dominant part of the singular integral equation which can be expressed as

$$\frac{1}{\pi}\int_{-h}^{h} G(t)\left[\frac{1}{t-x}+\frac{1}{2}\left\{4\lambda_2(h+x)^2\frac{d^2}{dx^2}+12\lambda_2(h+x)\frac{d}{dx}-\lambda_1+3\lambda_2\right\}\frac{1}{t-(2h+x)}\right.$$
$$\left.+\frac{1}{2}\left\{4\lambda_2(h-x)^2\frac{d^2}{dx^2}-12(h-x)\frac{d}{dx}-\lambda_1+3\lambda_2\right\}\frac{1}{t-(2h-x)}\right]dt$$
$$= -\frac{p(x)(1+\kappa_1)}{4\mu_1}, \qquad |x|<h \tag{24}$$

$G(t)$ is now assumed to have an integrable singularity at $t = \pm h$ which can be expressed as [10]

$$G(t) = \frac{H(t)}{(h^2-t^2)^\gamma} = \frac{H(t)\,e^{\pi i\gamma}}{(t-h)^\gamma(t+h)^\gamma}, \qquad |t|<h \tag{25}$$

where $0 < \mathrm{Re}(\gamma) < 1$ and $H(t)$ satisfies a Hölder condition in the closed interval $-h \le t \le h$. Technique of determining γ requires studying (24) near $t = \pm h$ [10, Chapter 4] and has been treated in detail in [5], [7] and [11]. Consider the following sectionally holomorphic function

$$\phi(z) = \frac{1}{\pi}\int_{-h}^{h}\frac{G(t)}{t-z}dt = \frac{1}{\pi}\int_{-h}^{h}\frac{H(t)\,e^{\pi i\gamma}\,dt}{(t-h)^\gamma(t+h)^\gamma(t-z)}. \tag{26}$$

According to ([10], Chapter 4)

$$\phi(z) = \frac{H(-h)}{(2h)^\gamma}\frac{e^{\pi i\gamma}}{\sin\pi\gamma}\frac{1}{(z+h)^\gamma}-\frac{H(h)}{(2h)^\gamma\sin\pi\gamma}\frac{1}{(z-h)^\gamma}+\phi_0(z) \tag{27}$$

where $\phi_0(z)$ is bounded everywhere except at the end points $\pm h$ where it has the following behavior

$$|\phi_0(z)| < \frac{H_0(\pm h)}{|z\pm h|^{\gamma_0}}, \qquad \mathrm{Re}(\gamma_0) < \mathrm{Re}(\gamma). \tag{28}$$

To consider (24) we need to reduce (27) for $z = x$, $2h+x$ and $2h-x$, obtaining

$$\phi(x) = \frac{H(-h)}{(2h)^\gamma}\frac{\cot\pi\gamma}{(h+x)^\gamma}-\frac{H(h)}{(2h)^\gamma}\frac{\cot\pi\gamma}{(h-x)^\gamma}+\phi^*(x), \qquad |x|<h$$
$$\phi(2h+x) = -\frac{H(h)}{(2h)^\gamma\sin\pi\gamma}\frac{1}{(h+x)^\gamma}+\phi_1^*(x), \qquad h<2h+x<3h \tag{29}$$
$$\phi(2h-x) = -\frac{H(h)}{(2h)^\gamma\sin\pi\gamma}\frac{1}{(h-x)^\gamma}+\phi_2^*(x), \qquad h<2h-x<3h.$$

Substituting (29) into (24) we obtain

$$\frac{1}{(2h)^{\gamma}\sin\pi\gamma(h+x)^{\gamma}}\left[H(-h)\cos\pi\gamma-\frac{H(h)}{2}\{4\lambda_2\gamma(\gamma+1)-12\lambda_2\gamma-\lambda_1+3\lambda_2\}\right]$$
$$-\frac{H(h)}{(2h)^{\gamma}\sin\pi\gamma(h-x)^{\gamma}}[\cos\pi\gamma+\tfrac{1}{2}\{4\lambda_2\gamma(\gamma+1)-12\lambda_2\gamma-\lambda_1+3\lambda_2\}] = P(x) \quad (30)$$

where $P(x)$ contains all the bounded functions.

Noting that for this solution of a symmetric problem $G(t)$ is an odd function, hence, $H(t)$ is odd, i.e. $H(t) = -H(-t)$, multiplying (30) by $(h+x)^{\gamma}$ and substituting $x = -h$ and then multiplying it by $(h-x)^{\gamma}$ and substituting $x = h$, we obtain the following characteristic equation for the determination of γ:

$$2\cos\pi\gamma+4\lambda_2(\gamma-1)^2-(\lambda_1+\lambda_2) = 0. \quad (31)$$

The equation is identical to the one obtained in [11]. The root of (31) satisfying $0 < \text{Re}(\gamma) < 1$ turns out to be a real constant for any material combination. If $a < h$ in Fig. 1, the only singular kernel will be the Cauchy kernel $1/(t-x)$. In that case, above analysis gives the characteristic equation as

$$\cot\pi\gamma = 0, \qquad \gamma = \tfrac{1}{2} \quad (32)$$

which is the well-known singularity at the crack tip in a homogeneous material away from any boundaries. Singular integral equation (17) now takes the following form.

$$\int_{-a}^{a}\frac{G(t)}{t-x}\,dt+\int_{-a}^{a}G(t)K_s(t,x)\,dt+\int_{-a}^{a}G(t)K_f(t,x)\,dt$$
$$= -\frac{\pi p(x)(1+\kappa_1)}{4\mu_1}, \qquad |x| < a, \quad a \le h \quad (33)$$
$$K_f(t,x) = \int_0^{\infty}[k(t,x,\eta)-k_{\infty}(t,x,\eta)]\,e^{-\eta(h-t)}\,d\eta$$

$K_s(t, x)$ given by (23) is a Fredholm's kernel for $a < h$ and becomes singular for $a = h$. $k(t, x, \eta)$ and $k_{\infty}(t, x, \eta)$ are given by equations (18) and (21) respectively and are such that $K_f(t, x)$ is a Fredholm's kernel for $a \le h$.

Solution of the integral equation

To solve (33), we first normalize the dimensions with respect to a, by introducing the following transformations

$$\tau = \frac{t}{a}, \qquad y = \frac{x}{a}, \qquad a_0 = \frac{a}{h}$$
$$G(t) = G(a\tau) = \phi(\tau), \qquad p(x) = p(ay) = s(y). \quad (34)$$

Hence (32) can be expressed as

$$\int_{-1}^{1}\phi(\tau)\left[\frac{1}{\tau-y}+aK_s(a\tau,ay)+aK_f(a\tau,ay)\right]d\tau = -\frac{\pi s(y)(1+\kappa_1)}{4\mu_1},$$
$$|y| < 1, \quad \frac{a}{h} = a_0 \le 1. \quad (35)$$

As indicated in the previous section, we can express $\phi(\tau)$ as

$$\phi(\tau) = \frac{\psi(\tau)}{(1-\tau^2)^\gamma} \tag{36}$$

where γ is given by (32) if $a_0 < 1$ and by (31) if $a_0 = 1$. (35) can be solved quite conveniently for $a_0 < 1$ by using the method developed in [6]. The method may be extended for $a_0 = 1$ as described in [5, 7, 11], where the Gauss–Jacobi integration formulas are used. Thus, (35) may be approximated by

$$\sum_{j=1}^{N} A_j\psi(\tau_j)\left[\frac{1}{\tau_j - y_i} + a\{K_s(a\tau_j, ay_i) + K_f(a\tau_j, ay_i)\}\right] = -\frac{\pi s(y_i)(1+\kappa_1)}{4\mu_1} \tag{37}$$

where

$$P_N^{(-\gamma,-\gamma)}(\tau_j) = 0, \qquad (j = 1, \ldots, N)$$

$$P_{N-1}^{(1-\gamma,1-\gamma)}(y_i) = 0, \qquad (i = 1, \ldots, N-1)$$

and A_j's are the corresponding weighting constants [5]. This provides us with $N-1$ linear algebraic equations for N unknowns $\psi(\tau_j)$, $j = 1, \ldots, N$. The additional equation is obtained by using the single valuedness condition (19):

$$\sum_{j=1}^{N} A_j\psi(\tau_j) = 0 \tag{38}$$

$\psi(\tau_j)$ are numerically evaluated from (37) and (38). We have the derivative of the crack surface displacement (11) as

$$\frac{\partial v_1}{\partial x}(x, 0) = G(x) = \frac{a^{2\gamma}H(x)}{(a^2-x^2)^\gamma}, \qquad |x| < a, \quad a \le h \tag{39}$$

where

$$H(x) = \psi\left(\frac{x}{a}\right).$$

Using the most common definition of the stress intensity factor as being the strength of the stress singularity ahead of the crack, we can write for the crack tip $x = a$ as

$$K = \lim_{x\to a} \sqrt{2}(x-a)^\gamma \sigma_{yy}^2(x, 0). \tag{40}$$

It can be shown [11] that

$$\lim_{x\to a,(x>a)} \sigma_{yy}^2(x, 0) = -2\mu^* \lim_{x\to a,(x<a)} \frac{\partial v_1}{\partial x}(x, 0) \tag{41}$$

where

$$\mu^* \begin{cases} = \dfrac{2\mu_1}{1+\kappa_1}, & a < h \\[2ex] = \dfrac{\mu_1\mu_2}{\sin\pi\gamma}\left[\dfrac{1+2\lambda_1(1-\gamma)}{\mu_1+\kappa_1\mu_2} + \dfrac{1-2\lambda_2(1-\gamma)}{\mu_2+\kappa_2\mu_1}\right], & a = h. \end{cases} \tag{42}$$

Hence the stress intensity factor K can be expressed as

$$K = -2\mu^*\left(\frac{a}{2}\right)^{\gamma}\sqrt{(2)}H(a) = -2\sqrt{(2)}\mu^*\left(\frac{a}{2}\right)^{\gamma}\psi(1). \tag{43}$$

NUMERICAL RESULTS AND DISCUSSION

The numerical results are obtained for a uniform pressure distribution on the crack surface. Also, since the case of $a < h$ has been treated previously* [1], results only for $a = h$ are presented. Thus the input function is

$$\sigma_{yy}^{1}(x, 0) = -p(x) = -p_0, \qquad |x| < h. \tag{44}$$

Figures 2 and 3 show the variation of the stress intensity factor with respect to the ratio of matrix-to-layer shear moduli μ_2/μ_1 for different Poisson's ratio combinations. These results are also presented in Table 1. From this table and Figs. 2 and 3, it is seen that the stress intensity factor ratio K/p_0h^{γ} decreases with a decrease in μ_2/μ_1. This value as $\mu_2/\mu_1 \to \infty$ corresponds to the semi-infinite strip fixed along $x = \pm h$, where the power of stress singularity and the stress intensity factor depend only on Poisson's ratio of the layer. As μ_1/μ_2 decreases, the power γ of the stress singularity increases, reaching a maximum when $\mu_2/\mu_1 = 0$ (i.e. when the layer is rigid). In this case, a non-integrable singularity

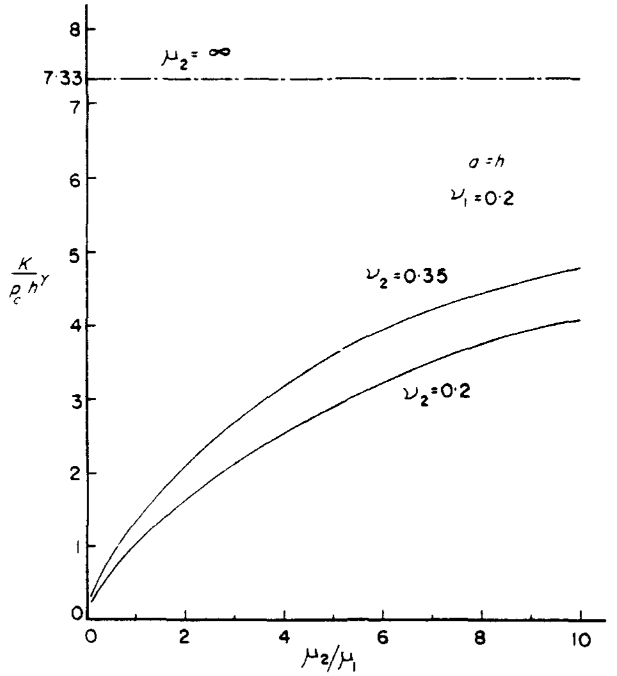

FIG. 2. Stress intensity factor vs. μ_2/μ_1. Layer Poisson's ratio = 0·2.

* Since a different method of solution was used in [1], spot check of a few points was made using the method presented here. The results were in good agreement with those in [1].

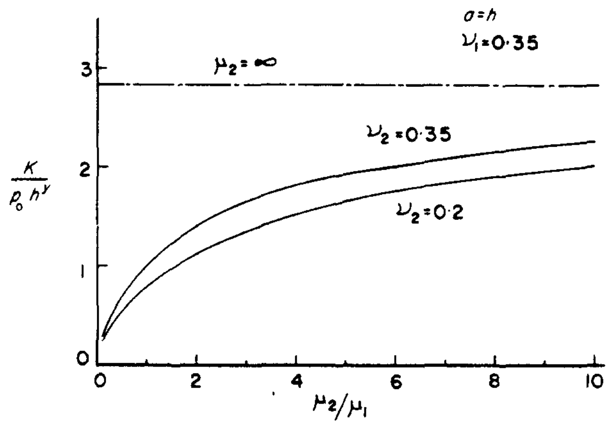

FIG. 3. Stress intensity factor vs. μ_2/μ_1. Layer Poisson's ratio = 0·35.

$\gamma = 1$ is obtained. Effect of Poisson's ratio on the stress intensity factor is also well depicted by these graphs and Table 1. Keeping the same Poisson's ratio for the layer and increasing it for the matrix, increases the stress intensity factor. A reverse effect is noted if we increase the Poisson's ratio of the layer and keep the matrix unchanged.

Table 1 also gives the results obtained by Ashbaugh [8] for $\nu_1 = \nu_2 = \frac{1}{3}$ and varying μ_2/μ_1. The numerical values may have some error since these are taken from the graphs provided in [8]. However, the results obtained by both methods are remarkably close. It is difficult to conclude which method gives more accurate results since both methods use numerical approximations. The main advantage of the method used in this paper is that the singular integral equation (33) can be solved assuming as if it were a Fredholm equation and a simple Gauss–Jacobi integration technique described in [7] yields good results.

Figure 4 shows the crack surface displacement for two sets of material combinations. The left ordinate scale corresponds to Epoxy–Aluminum combination. When the crack lies in epoxy layer, we observe much larger crack surface displacement in the center of the crack than when the aluminum layer contains the crack. Right ordinate scale corresponds to Aluminum–Steel combination in which the displacements are one order of magnitude smaller than Epoxy–Aluminum combination.

TABLE 1. STRESS INTENSITY FACTOR K/p_0h^γ

μ_2/μ_1	$\nu_1 = 0·2$		$\nu_1 = 0·35$		Results of Ref. [8]
	$\nu_2 = 0·2$	$\nu_2 = 0·35$	$\nu_2 = 0·2$	$\nu_2 = 0·35$	$\nu_1 = \nu_2 = \frac{1}{3}$
0·1	0·249	0·289	0·228	0·258	0·28
0·5	0·644	0·836	0·542	0·683	0·70
1·0	1·000	1·343	0·789	1·000	1·00
2·0	1·611	2·128	1·122	1·408	1·44
5·0	2·881	3·587	1·657	1·945	1·96
10·0	4·084	4·769	2·042	2·266	—
30·0	5·757	6·183	2·460	2·572	—
∞	7·332	7·332	2·831	2·831	2·90

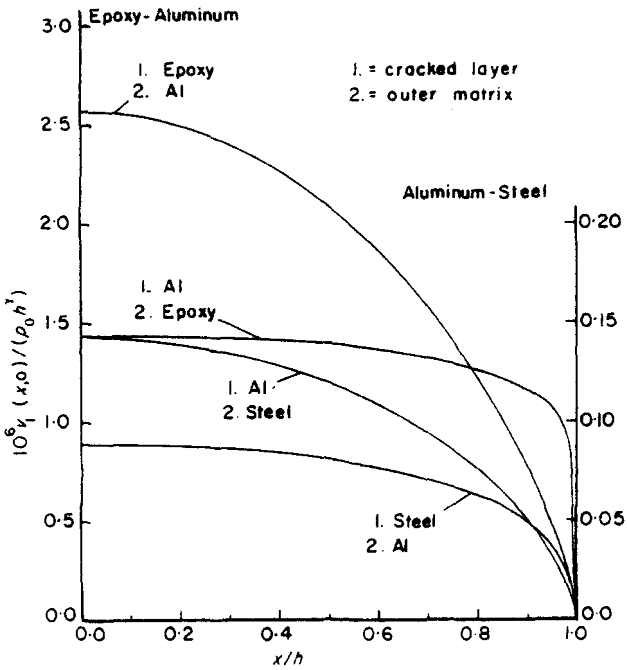

FIG. 4. Crack surface displacement for different material combinations.

Stress fields in both materials can be quite easily computed by using appropriate equations. Stresses in the neighborhood of the crack tip in such a case have been presented in [11]. Another question of interest is that if loads are increased, would the crack tend to propagate into the material 2, would it reflect back in 1 at some angle, or would debonding take place along the interface. A fracture criterion based on the "maximum stress" concept has been proposed in [11] and can very well be applied here. Validity of this criterion (or any other criterion) would have to be established by experimental studies.

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APPENDIX A

Consider the integrals given in (15). Substituting from (12) and changing the order of integration we obtain

$$\begin{aligned}
(\kappa_1+1)D_1(\eta) &= -2\int_0^a G(t)\,dt\,\frac{4\eta^2}{\pi}\int_0^\infty \frac{\xi\cos\xi h\sin\xi t}{(\xi^2+\eta^2)^2}\,d\xi \\
(\kappa_1+1)D_2(\eta) &= -2\int_0^a G(t)\,dt\,\frac{4\eta}{\pi}\int_0^\infty \frac{\xi^2\sin\xi h\sin\xi t}{(\xi^2+\eta^2)^2}\,d\xi \\
(\kappa_1+1)D_3(\eta) &= -2\int_0^a G(t)\,dt\,\frac{2\eta}{\pi}\int_0^\infty \left[\frac{2\eta^2}{(\xi^2+\eta^2)^2}+\frac{\kappa_1-3}{2}\,\frac{1}{\xi^2+\eta^2}\right]\sin\xi h\sin\xi t\,d\xi \\
(\kappa_1+1)D_4(\eta) &= -2\int_0^a G(t)\,dt\,\frac{2\eta^2}{\pi}\int_0^\infty \left[\frac{2\xi}{(\xi^2+\eta^2)^2}+\frac{\kappa_1+1}{2}\,\frac{1}{\xi(\xi^2+\eta^2)}\right]\cos\xi h\sin\xi t\,d\xi.
\end{aligned} \tag{A-1}$$

Using the properties of trigonometric functions and the fact that $G(t)$ is an odd function, the integrals can be reduced to the following form.

$$\begin{aligned}
(\kappa_1+1)D_1(\eta) &= \int_{-a}^a G(t)\,dt\,\frac{4\eta^2}{\pi}\int_0^\infty \frac{\xi\sin\xi(h-t)}{(\xi^2+\eta^2)^2}\,d\xi \\
(\kappa_1+1)D_2(\eta) &= -\int_{-a}^a G(t)\,dt\,\frac{4\eta}{\pi}\int_0^\infty \frac{\xi^2\cos\xi(h-t)}{(\xi^2+\eta^2)\eta}\,d\xi \\
(\kappa_1+1)D_3(\eta) &= -\int_{-a}^a G(t)\,dt\,\frac{2\eta}{\pi}\int_0^\infty \left[\frac{2\eta^2}{(\xi^2+\eta^2)^2}+\frac{\kappa_1-3}{2}\,\frac{1}{\xi^2+\eta^2}\right]\cos\xi(h-t)\,d\xi \\
(\kappa_1+1)D_4(\eta) &= \int_{-a}^a G(t)\,dt\,\frac{2\eta^2}{\pi}\int_0^\infty \left[\frac{2\xi}{(\xi^2+\eta^2)^2}+\frac{\kappa_1+1}{2}\,\frac{1}{\xi(\xi^2+\eta^2)}\right]\sin\xi(h-t)\,d\xi.
\end{aligned} \tag{A-2}$$

From the tables of Fourier Transform in [9] we have

$$\int_0^\infty \frac{1}{\xi(\xi^2+\eta^2)}\sin\xi y\,d\xi = \frac{\pi}{2\eta^2}(1-e^{-\eta y}) \tag{A-3}$$

$$\int_0^\infty \frac{\xi}{(\xi^2+\eta^2)^2}\sin\xi y\,d\xi = \frac{\pi}{4\eta}y\,e^{-\eta y}. \tag{A-4}$$

Therefore, by appropriate differentiation on both sides, we obtain

$$\begin{aligned}
\int_0^\infty \frac{1}{\xi^2+\eta^2}\cos\xi y\,d\xi &= \frac{\pi}{2\eta}e^{-\eta y} \\
\int_0^\infty \frac{\xi^2}{(\xi^2+\eta^2)^2}\cos\xi y\,d\xi &= \frac{\pi}{4\eta}(1-\eta y)\,e^{-\eta y} \\
\int_0^\infty \frac{\eta^2}{(\xi^2+\eta^2)^2}\cos\xi y\,d\xi &= \frac{\pi}{4\eta}(1+\eta y)\,e^{-\eta y}.
\end{aligned} \tag{A-5}$$

Using (A–3)–(A–5) with $y = h-t$ in equations (A–2), we obtain the desired result in (16) as

$$\begin{aligned}
(\kappa_1+1)D_1(\eta) &= \int_{-a}^{a} G(t)\eta(h-t)\,e^{-\eta(h-t)}\,dt \\
(\kappa_1+1)D_2(\eta) &= \int_{-a}^{a} G(t)[\eta(h-t)-1]\,e^{-\eta(h-t)}\,dt \\
(\kappa_1+1)D_3(\eta) &= -\int_{-a}^{a} G(t)\left[\eta(h-t)+\frac{\kappa_1-1}{2}\right]e^{-\eta(h-t)}\,dt \\
(\kappa_1+1)D_4(\eta) &= \int_{-a}^{a} G(t)\left[\eta(h-t)-\frac{\kappa_1+1}{2}\right]e^{-\eta(h-t)}\,dt+\int_{-a}^{a} G(t)\,dt.
\end{aligned} \tag{A–6}$$

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Абстракт—Исследуется задача составных слоистых пластиков, при налиуии трещины, расположенной нормально к линиям сцепления. Дается вывод анализа напряжений для предельного случая, когда трещина распространяется к линиям сцепления. Применяется метод интегрального преобразования, в целью формулировки задауи, выраженной сингулярным интегральным уравнением. Из этого уравнения получается степень особенности напряжений вокруг конца трещины, присоединенной к поверхностям раздела. Решается численно сингулярное интегральное уравнение. Подсчитывается эффект свойств материала, в зависимости от фактора интенсивности напряжений.